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5 Spatial Skills, Reasoning, and Mathematics

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Modern technological societies are built on a foundation of mathematics. We could not have extensive trade without book-keeping – we would be stuck with a barter system. We could not build our long bridges without calculation – we would still be relying on ferries to cross bodies of water. We have made impressive improvements in agricultural science in the past century based in part on experiments using the statistics of "split plots." The examples could be multiplied but the lesson is clear. Given the importance of numeracy, there is good reason for educational systems to strive to teach mathematics effectively. Even though many children in contemporary schools succeed in learning to calculate, many others struggle or progress slowly, and even more never achieve the levels required for full participation in our technological society. There are many reasons for this situation and many proposed remedies. One potential way to improve mathematics education involves harvesting our growing understanding of how human minds and brains process quantitative information and how these processes develop. The teaching of reading has already benefited from the insights of cognitive science (Rayner et al., 2001; Castles, Rastle, & Nation, 2018) and the teaching of mathematics is starting to keep pace (Ansari & Lyons, 2016).

The purpose of this chapter is to evaluate the potential of leveraging mathematics learning based on the links between spatial thinking and mathematical learning. A few sample findings give some sense of the variety of the evidence, which is derived from many levels of analysis. At the neural level, for example, Amalric and Dehaene (2016) found a great deal of overlap between the brain areas used for spatial and mathematical processing, even in expert mathematicians and across a wide range of mathematical fields. In terms of development, spatial–numerical associations are apparently basic, present at birth and even shared with other species, although also modified by culture (Rugani & de Hevia, 2017). Behaviorally, there is a longitudinal link between spatial skills and mathematical achievement, evident as young as preschool and continuing into high school and university (Casey et al., 1995; Kyttälä et al., 2003; Shea, Lubinski, & Benbow, 2001; Verdine et al., 2017; Wai, Lubinski, & Benbow, 2009).

Thus, one hope is that improving spatial skills will improve mathematics achievement.

This strategy would benefit, however, from delineating the pathways linking spatial skills to numeracy skills. We know that spatial skills and numeracy skills are both multidimensional constructs (Mix & Cheng, 2012; Uttal et al., 2012), so specific spatial skills may underlie specific mathematical achievements, in which case intervention should focus on the relevant spatial skills. Alternatively or in addition, mathematics learning may be facilitated overall by a general spatial way of thinking, what has been called a *spatial turn of mind*. For example, children and adults with the habit of visualizing problems, or perhaps even actually sketching them, may find mathematical reasoning easier. We discuss what we know about the nature of spatial–mathematical linkages in two major sections, concentrating first on elementary school mathematics and then on secondary school mathematics.

Spatial–Mathematical Linkages in Preschool and Elementary School

Understanding spatial–mathematical linkages requires some understanding of the nature of development of both domains. In this section, we begin with a short summary of early mathematical development and then turn to three kinds of spatial processes that may be relevant to mathematical development in this age range. One process is visuospatial working memory (VSWM), which is arguably a resource more than a skill. As we turn to skills, although there are a variety of spatial skills, only some have been extensively investigated. In this section, we concentrate on mental rotation and on proportional reasoning/spatial scaling. Proportional reasoning and spatial scaling have been studied separately but turn out to have a great deal in common. We close with a consideration of spatial strategies, the more general way in which spatial thinking may influence mathematical reasoning.

The Nature of Early Mathematical Learning

One important distinction for young children is between symbolic approximation skills (sometimes referred to as “number sense”) and exact numeracy skills. Symbolic approximation skills involve rapidly estimating relations between symbolic quantities (e.g., approximate symbolic calculation and numerical comparison) and are thought to rely on a mapping between the evolutionarily-old, nonsymbolic approximate number system (ANS) and a set of culturally created symbolic representations (number words and Arabic numerals) (Carey, 2009; Feigenson, Dehaene, & Spelke, 2004). Both symbolic and nonsymbolic approximate number representations become more fine-tuned with age and education (Halberda & Feigenson, 2008; Sekuler & Mierkiewicz, 1977), allowing adults to make faster and more precise judgments about numerical quantity than young children. Interestingly, approximation tasks involve activation in the intraparietal sulcus (IPS) in children and adults (Halberda & Feigenson, 2008; Kaufmann et al., 2011), a region also implicated in mental rotation skill (Zacks, 2007).

In contrast, exact numeracy skills involve concepts and procedures necessary to precisely represent and manipulate quantities (e.g., cardinality and exact arithmetic). These exact numeracy skills, which for young children involve whole-number concepts and procedures, are thought to rely on different processes and neural substrates than approximate numeracy skills. Children ages 3–5 who are just learning the cardinal meanings of the first few count words (i.e., "one," "two," and "three") appear to map them onto an object-based representation system that can hold up to three or four items in memory (Carey, 2009) and only later map them to approximate representations in the ANS (Le Corre, 2014; Le Corre & Carey, 2007). In adults, performing exact calculations through direct retrieval involves activation of the left angular gyrus (Grabner et al., 2009; Grabner et al., 2007), which is close to language processing areas but distinct from the IPS, the area that is implicated in approximation tasks. Thus, the neural data suggest that exact calculation in adults is supported by verbal processes.

In addition to skills involving approximate calculation, number comparison, counting, and exact calculation, another important numerical representation that develops in childhood is the number line. Humans are predisposed to associate spatial magnitude (such as line length or area) with numerical magnitude, even in the absence of formal schooling (Dehaene, Bossini, & Giraux, 1993; Dehaene et al., 2008; de Hevia & Spelke, 2010; Lourenco & Longo, 2010; Pinel et al., 2004; Zorzi, Priftis, & Umiltà, 2002). Children in Western societies begin to map symbolic numbers (Arabic numerals) to space in a left-to-right orientation as early as preschool and kindergarten (Ebersbach, 2015; Sella et al., 2017). Theoretical accounts describe children's number line representations as initially logarithmic, in which they allocate more space to smaller numbers and less space to larger numbers (Opfer & Siegler, 2007; Opfer, Siegler, & Young, 2011; Siegler & Opfer, 2003). However, with age and experience, children's number line representations shift toward greater linearity, such that numbers that are equally distant in terms of numerical magnitude are represented in a spatially equidistant manner (Booth & Siegler, 2006; Siegler, 2009; Siegler & Booth, 2004; Siegler & Booth, 2005; Siegler & Opfer, 2003). Developing a linear number line representation may involve narrowing the neural "tuning curves" associated with each Arabic numeral in the IPS (for a review, see Kaufmann et al., 2011), so that the amount of representational overlap between successive numbers is similar regardless of the size of the numbers.

Although it seems clear that mature performance on a typical number line task involves proportional judgments about a number in relation to the number line's endpoints (Slusser, Santiago, & Barth, 2013), there is controversy regarding how to best describe the earlier, logarithmic (or pseudo-logarithmic) stage (Barth & Paladino, 2011; Barth et al., 2011; Ebersbach et al., 2008; Kim & Opfer, 2017; Moeller et al., 2009; Opfer, Thompson, & Kim, 2016). Despite this controversy, there is strong evidence that the accuracy of children's number line estimations is a strong predictor of other numeracy skills, including numerical magnitude comparison, number recall, approximate calculation, and symbolic estimation (Booth & Siegler, 2008; Laski & Siegler, 2007; Siegler & Ramani, 2008, 2009). Further, lessons incorporating number lines are effective for teaching children concepts and procedures related to whole numbers and fractions (Fuchs et al., 2013; Hamdan & Gunderson, 2016; Saxe, Diakow, & Gearhart, 2013).

Visuospatial Working Memory

VSWM is one component of the three-part model of working memory (central executive, phonological loop, and VSWM) proposed by Baddeley and Hitch (1974). It is thought to store and process information in terms of its visual and spatial features. VSWM undergoes substantial development in early childhood (Gathercole et al., 2004) and is a robust predictor of numeracy skills in pre-k through 3rd grades. In prekindergarten, VSWM predicts counting skills (Kyttälä et al., 2003) and non-verbal addition (Rasmussen & Bisanz, 2005). In kindergarten, VSWM correlates with number line estimation (0–100), rapid identification of groups that add to 5 (Geary et al., 2007), and arithmetic performance (McKenzie, Bull, & Gray, 2003; Xenidou-Dervou, van der Schoot, & van Lieshout, 2015). In 1st grade, experimental disruption of VSWM using a dual task harms arithmetic performance (McKenzie et al., 2003). In 2nd and 3rd grades, VSWM relates to calculation skills (Nath & Szücs, 2014) and general math achievement (Gathercole & Pickering, 2000; Meyer et al., 2010). These relations are not only concurrent but also predictive. In one study, VSWM at age five predicts general math achievement in 3rd grade, mediated by quantity-number competencies at age six (Krajewski & Schneider, 2009). In another longitudinal study, four-year-olds' VSWM predicted growth in calculation skills over a fourteen-month period, even after accounting for vocabulary, processing speed, and nonsymbolic numerical discrimination skills (Soto-Calvo et al., 2015).

VSWM, as a versatile "mental visual sketchpad," may impact multiple aspects of numeracy that rely on this capacity, both approximate and exact. In a study of kindergarteners completing single-digit calculations, VSWM, but not verbal working memory, related to approximate symbolic calculation skills, and verbal working memory, but not VSWM, related to exact symbolic calculation skills (Xenidou-Dervou et al., 2015). Although exact calculations, particularly those involving rote memorization, rely heavily on verbal processes (e.g., Grabner et al., 2007; Spelke & Tsivkin, 2001), other strategies for exact calculation rely more heavily on VSWM. For example, VSWM may help children keep track of objects while counting and visualize nonsymbolic "mental models" of simple arithmetic problems (e.g., visualizing 2 objects and 3 objects to compute 2 + 3) (Alibali & DiRusso, 1999; Geary et al., 2004; Huttenlocher, Jordan, & Levine, 1994). Further, mentally computing a multistep symbolic calculation requires VSWM to remember intermediate steps involving carry operations and place value, especially when problems are presented vertically (Caviola et al., 2012; Trbovich & LeFevre, 2003). In addition, VSWM may help children to remember and later hold in mind the number line representation, which may, in turn, foster other numeracy skills (e.g., Booth & Siegler, 2008; Siegler & Ramani, 2008).

Mental Rotation

Mental rotation is the ability to hold in mind and mentally rotate representations of 2-D or 3-D visual stimuli (e.g., to decide whether a rotated puzzle piece would fit into a puzzle) (Shepard & Metzler, 1971). Mental rotation has been found to predict

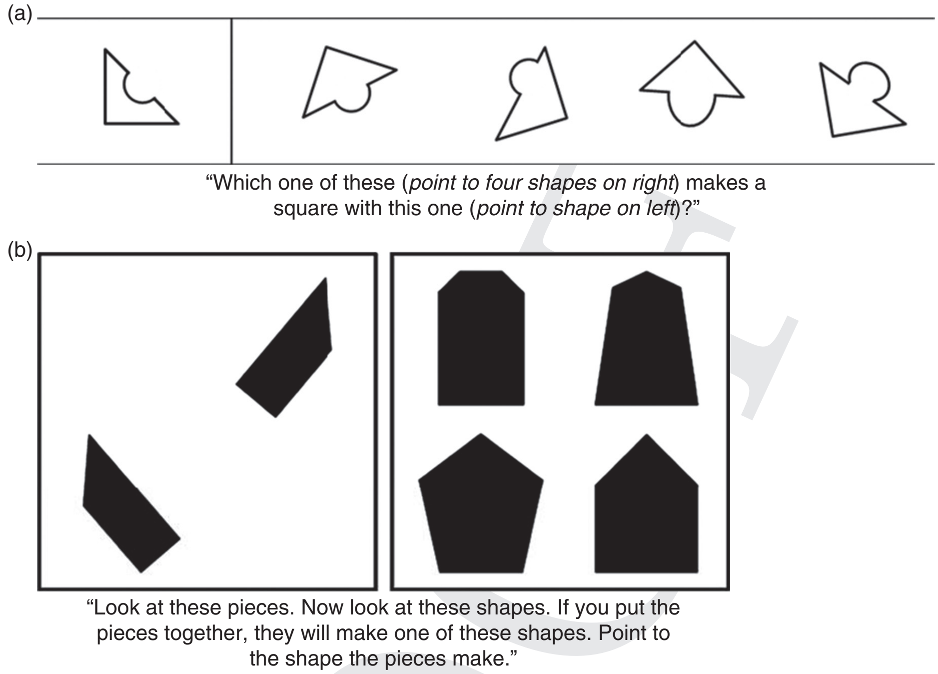


Figure 5.1 *(a) Illustration of the type of items used on the Thurstone mental rotation task; (b) A sample item from the Children's Mental Transformation Task (from Gunderson, et al., 2012, p. 1233. Reprinted with permission from the American Psychological Association)*

several measures of numeracy in young children (pre-k to 4th grade) using age-appropriate tasks (Figure 5.1). In pre-k, mental rotation skill correlates with a composite of numeracy skills (including counting, cardinality, number comparison, and ordering) (Kyttälä et al., 2003). Among 1st grade girls, mental rotation skill correlates with arithmetic proficiency (Casey et al., 2014). Further, mental rotation skills in 1st and 2nd grade predicted growth in number line knowledge over the course of the school year (Gunderson et al., 2012).

In a separate sample, mental rotation skills at age five predicted approximate symbolic arithmetic performance at age eight. The strongest correlational evidence to date shows that mental rotation skills uniquely relate to kindergarten and 3rd graders' (but not 6th graders') concurrent math skills (measured as a single factor), even after controlling for a variety of other spatial skills (Mix et al., 2016). Finally, one experimental study found that experimentally training mental rotation skill yielded improvements in arithmetic among six- to eight-year-olds, especially on missing-term problems (Cheng & Mix, 2012), although one attempt to replicate this effect of mental rotation training on numeracy was unsuccessful (Hawes et al., 2015). However, encouragingly, several recent randomized studies using more varied spatial training regimes, including mental rotation as well as other spatial skills (often in a playful context), have found positive effects of spatial training on

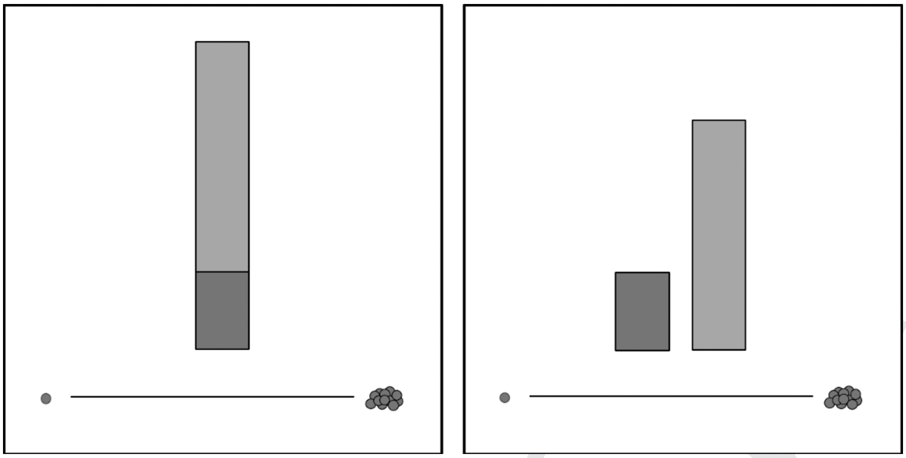


Figure 5.2 *Proportional reasoning measure*

numerical skills in young children (Grissmer et al., 2013; Hawes et al., 2017; Lowrie, Logan, & Ramful, 2017). Taken together, the research robustly supports a correlation between mental rotation skill and numeracy. The causal impact of spatial training (in general) on numeracy is increasingly well-supported, although evidence for transfer from mental rotation training (in particular) to numeracy is more mixed.

Despite the strong correlation between mental rotation skills and multiple aspects of numeracy, the mechanisms through which mental rotation skill would affect numeracy are not obvious. One mechanism, proposed by Cheng and Mix (2012), is that mental rotation skills help children to “rotate” missing-term arithmetic problems (e.g., 1 + __ = 5) into a more conventional format (e.g., __ = 5 – 1). Another possibility is that mental rotation skills are one component of a broader skill of spatial visualization – the ability to manipulate mental representations of objects in space – and that this spatial visualization skill can be brought to bear in a variety of numerical contexts that involve grounding new or complex concepts in a spatial mental model. Consistent with this hypothesis, block design (another measure of spatial visualization) was uniquely related to newly learned math concepts, but not familiar math concepts, among children in kindergarten, 3rd, and 6th grades; mental rotation was related to both new and familiar concepts among kindergarten and 3rd graders (Mix et al., 2016).

Proportional Reasoning and Spatial Scaling

Proportional reasoning and spatial scaling are two spatial skills that have only recently begun to be investigated in terms of individual differences (Frick & Newcombe, 2012; Möhring et al., 2014; Möhring et al., 2016). Proportional reasoning involves understanding part–whole or part–part relations between spatial extents (see Figure 5.2); spatial scaling involves reasoning about a representation that differs in size from its referent (e.g., a map that differs in size from the city it represents).

These skills are deeply related: Proportional reasoning requires recognizing equivalent proportions at different scales (e.g., 2 cm of a 10 cm line is proportionally equivalent to 20 cm of a 100 cm line), and spatial scaling involves using proportional information to map locations between scales (e.g., a location on a map that is one-third of the way between two buildings will also be one-third of the way between those buildings at full scale). Indeed, recent work indicates that proportional reasoning (using nonsymbolic spatial extents) and spatial scaling are significantly correlated in childhood (Möhring, Newcombe, & Frick, 2015).

Although classic work by Piaget and Inhelder (1975) argued that proportional reasoning skill did not emerge until around eleven years of age, more recent work has shown sensitivity to proportions even in infancy (Duffy, Huttenlocher, & Levine, 2005; Huttenlocher, Duffy, & Levine, 2002). Starting at six months of age, infants and young children are quite sensitive to proportional relations between spatial extents (e.g., lengths), while the ability to discriminate exact spatial extents emerges much later, after age four (Duffy et al., 2005; Huttenlocher et al., 2002). Both children and adults spontaneously use proportional strategies, biased toward the center of salient spatial categories (such as quadrants of a circle or number line), to remember locations and make explicit proportion judgments related to 1-D and 2-D spaces (Huttenlocher, et al., 2004; Huttenlocher, Hedges, & Vevea, 2000; Huttenlocher, Newcombe, & Sandberg, 1994; Spence & Krizel, 1994). Spatial scaling ability also develops early: Children ages 3–6 show individual differences in the ability to use a 2-D map to find a location in another 2-D space that differs in size (Frick & Newcombe, 2012; Möhring et al., 2014; Vasilyeva & Huttenlocher, 2004).

Work on proportional reasoning and scaling is relatively new, and their links to numeracy are less well-tested than for mental rotation and VSWM. One recent study has shown that proportional reasoning skill is correlated with symbolic fraction concepts in children ages 8–10 (Möhring et al., 2016). Further, there are strong theoretical reasons to believe that proportional reasoning should relate to number line knowledge, since the number line also requires estimating quantities and relating parts to wholes. In fact, there is evidence that mature performance on a symbolic number line task involves proportion judgments that are biased toward the center of salient categories (such as halves or quarters of the number line), similar to proportion judgments in nonsymbolic, visual tasks (Barth & Paladino, 2011; Barth et al., 2011). Consistent with this, 5th graders' nonsymbolic proportional reasoning skills loaded onto the same factor as number line estimation (Ye et al., 2016). If proportional reasoning helps children's number line estimation, this may in turn benefit their numeracy skills more broadly.

The Linear Number Line

Despite decades of research showing a correlation between spatial skills and numeracy, relatively little work has probed the mechanisms that might explain this link. One potential mechanism is that the acquisition of a specific cultural tool that brings together spatial and numerical representations – *the linear number line* – may help to explain the relation between spatial skills and symbolic numeracy skills (Gunderson et al., 2012). Spatial skills may facilitate the development (i.e., learning, retention, and use) of the linear number line representation, which in turn enhances other numeracy skills, especially those related to symbolic approximation (Gunderson et al., 2012). Gunderson et al. (2012) reported on two longitudinal studies supporting this theory. In the first study, 1st and 2nd graders' beginning-of-year mental rotation skills predicted improvement in number line knowledge over the course of the school year, even after accounting for beginning-of-year math and reading achievement. In the second study, children's mental rotation skills at age five predicted approximate symbolic calculation ability at age eight, mediated by number line knowledge at age six. A separate study of 2nd through 4th graders replicated and extended these results, finding that the longitudinal relation between spatial skills and later calculation skill was partially mediated by number line knowledge (LeFevre et al., 2013).

Mental rotation may contribute to a visual transformation strategy (e.g., zooming) during number line estimation. Additional spatial skills may also be involved. As noted previously, proportional reasoning (Ye et al., 2016) and VSWM (Geary et al., 2007) have also been linked to number line estimation skill. Proportional reasoning may contribute to a proportion judgment strategy, and VSWM may help children to recall locations on number lines they have encountered in school. However, because the relations of mental rotation, proportional reasoning, and VSWM to number line knowledge have been investigated in separate studies, more work is needed to determine whether all three skills contribute uniquely to children's number line knowledge. Despite these limitations and open questions, research to date is consistent with the hypothesized causal chain linking spatial skills to number line knowledge to symbolic calculation skills, both exact (LeFevre et al., 2013) and approximate (Gunderson et al., 2012).

In terms of its relation to numeracy, one theoretical possibility is that the number line representation is particularly helpful for approximate symbolic numeracy skills, to the extent that improvement on the number line task indicates more finely tuned magnitude representations that are especially critical for approximation. Indeed, many studies showing the impact of number line estimation on numerical skill have used approximate measures (Booth & Siegler, 2008; Gunderson et al., 2012; Laski & Siegler, 2007; Siegler & Ramani, 2008, 2009). However, even if it is especially important for approximation, number line estimation skill may impact exact symbolic numeracy skills as well, perhaps by increasing children's ability to notice and correct errors in exact calculation procedures. Consistent with this, recent studies have also shown a strong relation between number line estimation and exact calculation skill (LeFevre et al., 2013; Xenidou-Dervou et al., 2015).

Spatial Strategy Use

Another potential mechanism linking spatial skills and numeracy is the use of spatial strategies (i.e., use of explicit visualization or external representations, such as schematic spatial images or sketches) to represent and solve a math problem. Use of spatial strategies is related to both spatial skills and math achievement (Blazhenkova, Becker, & Kozhevnikov, 2011; Hegarty & Kozhevnikov, 1999), which gives reason to believe that spatial strategy use may mediate the relation between spatial skills and numeracy skills. Children as young as age eight, as well as adults, can reliably self-report their preference for the use of spatial visualization strategies, object visualization strategies (i.e., detailed pictorial images of objects in the relevant problem), and verbal strategies (Blajenkova, Kozhevnikov, & Motes, 2006; Blazhenkova et al., 2011). Among older children (ages 8–18), spatial strategy preference is significantly related to children's mental rotation skill and relates to children's intention to pursue STEM fields (physics, chemistry, math, and computer science) (Blazhenkova et al., 2011). In addition, children's actual use of spatial strategies while completing math word problems predicts success on those problems (Hegarty & Kozhevnikov, 1999). Thus, children with higher levels of spatial skills may be more likely to use spatial strategies while completing numerical tasks (especially novel or difficult ones), leading to improved performance. However, given the paucity of research in this area, it may be fruitful for researchers to investigate the relations between specific spatial skills (such as mental rotation, VSWM, and proportional reasoning), spatial strategy preference and use, and math achievement among young children.

Spatial–Mathematical Linkages in Secondary School

Much like findings for primary mathematics content, there is evidence for the influence of spatial skills on higher-level mathematics skills, such as those learned in secondary school. Both VSWM and mental rotation (but not verbal working memory) are predictive of higher-level mathematics achievement scores (Reuhkala, 2001), and 3-D spatial visualization tasks such as mental rotation and paper folding have been found to predict students' SAT-M scores as they exit secondary school, as well as to mediate the observed relation between verbal working memory and SAT-M scores (Tolar, Lederberg, & Fletcher, 2009). Correlations between these types of 3-D spatial visualization measures and mathematics achievement tend to be greater for higher-level mathematics skills than for elementary mathematics skills (Casey, Nuttall, & Pezaris, 1997; Friedman, 1995; Reuhkala, 2001).

In the following sections, we first describe the types of mathematical content studied in secondary schools and then describe evidence for relations between spatial skills and these various areas of higher-level mathematics. We then discuss the mechanisms that may explain the connections between spatial reasoning and these higher-level mathematics skills.

The Nature of Secondary Mathematical Learning

In secondary schools, mathematics learning typically encompasses three types of content through which students progress at different speeds and to different degrees. Secondary mathematics often begins with the study of algebra in middle or early high school. Algebra focuses on the understanding of variables and their operations (Usiskin, 1988) and generally contains component skills such as solving equations and word problems and graphing linear, quadratic, and exponential functions. The next facet of secondary mathematics is typically geometry, which is typically studied in a stand-alone course in high school in the United States, but significant content on 2-D and 3-D figures is also addressed in middle school. Geometry has been defined as "branches of mathematics that exploit visual intuition . . . to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society and JMC, 2001). Finally, students who are successful in these earlier mathematics courses may begin the study of calculus, which is typically restricted to advanced high school and college students. Calculus can be thought of as the study of change in mathematical quantities, typically encompassing knowledge of derivatives, integrals, and limits (Zuccheri & Zudini, 2014).

Perhaps because they are studied by most secondary students, algebra and geometry topics dominate the literature on secondary mathematics learning. However, there are surprisingly few studies on connections between spatial skills and algebra (Kinach, 2012).

Compared with algebra, a greater amount of research has established the link between spatial ability and geometry (e.g., Battista, Wheatley, & Talsma, 1982, Delgado & Prieto, 2004). Unfortunately, there is a relative dearth of research on learning calculus at all, much less on the role of spatial abilities.

Here, we review the extant literature on two types of spatial skills: VSWM and mental rotation. Where available, we include findings on how spatial skills are related to all three of these types of secondary mathematics content; though geometry may be considered the most obvious example of how spatial reasoning is relevant to secondary mathematics, mathematicians argue that "much of the thinking that is required in higher mathematics is spatial in nature" (Jones 2001, p. 55).

Visuospatial Working Memory

Since 2008, several studies have examined potential connections between various higher-level mathematics skills and VSWM. For instance, VSWM was shown to predict Australian high school students' ability to solve symbolic algebraic problems (Trezise & Reeve, 2014). Kyttälä and Lehto (2008) also found a direct relation between VSWM and performance on algebraic word problems for Finnish high school students; however, they did not find a comparable relation between VSWM and geometry problem-solving in that population.

Even in cases where such a relation is found, the link between VSWM and geometry achievement appears to be very weak and perhaps limited only to tasks involving mental manipulation (Giofrè et al., 2013). Using Dehaene and colleagues' (2006) distinction between types of geometric principles, Giofrè and colleagues (2013) also found that the relation between VSWM and geometry was limited to culturally mediated principles of geometry (i.e., symmetry, chirality, metric properties, and geometrical transformations) and not to core principles of geometry (i.e., topology, Euclidean geometry, and geometric figures; see Figure 5.3).

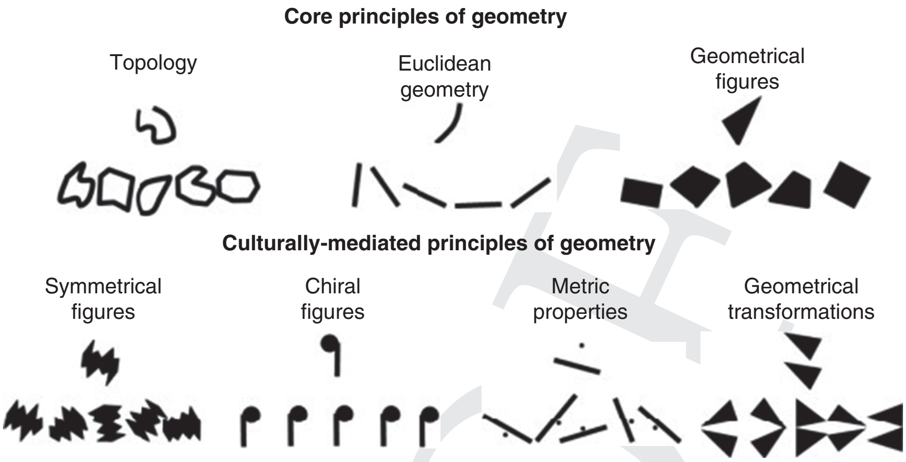


Figure 5.3 *Examples of core vs. culturally mediated principles of geometry (from Giofrè, et al., 2013, p. 117. Copyright 2013 by Elsevier. Reprinted with permission)*

In a recent meta-analysis, Peng and colleagues (2015) aimed to compare the impact of different facets of working memory on various aspects of secondary mathematics learning. They concluded that the role of working memory in geometry performance was generally small and that VSWM was no more influential than any other type of WM. They did not draw conclusions regarding VSWM and other facets of secondary mathematics, however, due to an insufficient number of studies on VSWM in algebra or any type of working memory at all in calculus.

Mental Rotation

Compared with those for VSWM, studies on the links between mental rotation (or other sorts of 3-D spatial visualization) and higher-level mathematics have yielded more conclusive findings. In general, much stronger evidence exists linking 3-D spatial visualization to geometry compared with algebra (Battista, 1990, Delgado & Prieto, 2004). For instance, Kyttälä and Lehto (2008) found a direct relation between mental rotation and solving geometry problems but only an indirect relation between mental rotation and solving algebraic word problems. High school students' performance on the Mental Rotation Test (MRT) has been found to relate to both geometry course grades and performance on a geometry achievement test, as well as to the students' perceptions of how well they do in geometry (Weckbacher & Okamoto, 2014); Mental rotation was also positively related to the students' perceptions of how well they do in algebra but not to their actual algebra course grades (Weckbacher & Okamoto, 2014). One recent study, however, found that scores on a paper folding task were predictive of both algebra pattern knowledge and geometry problem-solving for 6th grade students in Singapore (Logan, 2015).

Three-dimensional spatial visualization has also been linked to calculus performance. Cromley and colleagues (2017) demonstrated a relationship between mental rotation and high school and college calculus students' scores on items from the AP calculus test, though mental rotation was not related to performance on a conceptual calculus measure. Similarly, Samuels (2010) found a significant correlation between scores on the Purdue Spatial Visualization Tests (PSVT) Development (3-D paper folding) task and solution of problems involving finding the derivative in a college calculus class. There is also some causal evidence of the relation, as practice reasoning about the rotation of 3-D objects led to improved calculus grades for low-spatial undergraduate engineering students (Sorby et al., 2013).

Pittalis and Christou (2010) isolated the effects of a composite spatial abilities measure (which included both mental rotation and paper folding tasks but not VSWM) across the board for four separate types of geometry reasoning (see Figure 5.4) in Cyprian middle-grade students; strong relations were found between spatial abilities and students' representations of 3-D objects and measurement skills with 3-D objects (i.e., calculating surface area and volume), while slightly weaker relations were found with spatial structuring tasks (i.e., arranging and enumerating unit cubes) and conceptualizing properties of 3-D shapes. A different composite spatial ability measure (that still included both mental rotation and paper folding tasks) was related not only to Canadian 7th and 8th graders' initial knowledge about representation of 3-D geometric shapes but also to how much they were able to learn from geometry instruction (Kirby & Boulter, 1999). However, only a weak link between a 3-D spatial visualization composite (mental rotation and paper folding) and algebraic equation solving was found in undergraduate students (Tolar et al., 2009).

Mechanisms for Explaining Spatial–Achievement Relations in Secondary Mathematics

In algebra, VSWM seems to be more influential than 3-D spatial visualization, whereas the opposite may be true in geometry and calculus. The effect of 3-D spatial visualization seems to manifest across the board for geometry content but only certain facets of algebra and calculus may be influenced by spatial skills. Why might particular facets of spatial ability influence particular domains (or subcomponents of domains) of mathematics achievement?

Perhaps the most obvious mechanism is that many mathematical domains are inherently spatial. Geometry involves working with two and three dimensional figures, and both algebra and calculus involve working with lines or curves on Cartesian graphs. Perhaps the reason why mental rotation and other forms of 3-D spatial visualization are especially influential in the domains of geometry and calculus is that the spatial aspects of the requisite mathematics are not static. In geometry, students learn about their invariance, symmetry, and *transformations* of shapes (Jones, 2002). Calculus is similarly about transformations – to find an integral, one must consider the cumulative area of rectangles under a curve over a range of x values; finding a derivative requires considering how the slope of the

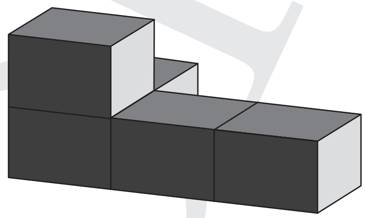
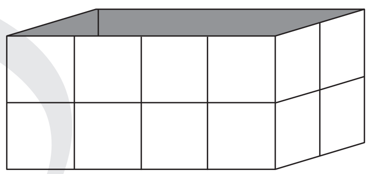
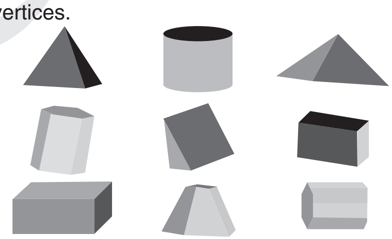
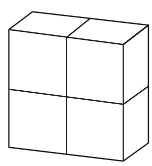
Ability	Description of Tasks	Example
Recognition and construction of nets	1. Identification of cuboids nets 2. Construction of a cylinder net 3. Construction of a triangular prism net 4. Identification of pyramid nets	Complete the following net in a proper manner to construct a triangular prism when folded.
Manipulation of 3D shapes representation modes	5. Translation of an orthogonal view to isometric 6. Translation of a side projection view to an orthogonal one 7&8. Recognition of parallel/perpendicular edges of a cube drawn in an isometric view 9&10. Enumeration of the triangular faces of a triangular pyramid/prism drawn in a transparent view	Draw the front, side and top view of the object. 
Structuring 3D arrays of cubes	11. Enumeration of the cubes needed to transform an object to a cuboid 12. Enumeration of the cubes and cuboids that fit in a box (the box is not empty) 13. Enumeration of the cubes that fit in an open/not empty box 14&15. Enumeration of the cubes that fit in an empty box	How many unit-sized cubes can fit in the box? 
Recognition of 3D shapes' properties	16. Recognition of cuboids 17. Recognition of solids that have a specific number of vertices 18,19&20. Enumerating the vertices/faces/edges of pyramids	Circle the solids that have at least 8 vertices. 
Calculation of the volume and the area of solids	21. Calculation of the area of a solid constructed by unit-sized cubes 22&23. Calculation of the area /volume of cuboids presented as open nets 24. Comparing the capacity of rectangular and cylinder reservoirs.	How much paper is needed to wrap the box? 
Comparison of 3D shapes properties	25. Right/wrong statements referring to the elements and properties of three solids 26&27. Exploration of the Euler's rule in pyramids/extension in prisms	Which of the following statements are wrong? (a) The faces of prisms and cuboids are rectangles, (b) the base of prisms and cuboids could be a rectangle and (c) the base of prisms and cuboids could be a triangle

Figure 5.4 *Classifications of spatial abilities (from Christou, 2010, p. 209. Copyright 2010 by Springer Science+Business Media B.V. Reprinted fwith permission)*

tangent to the curve changes as the x value changes (Bremigan, 2005; Sorby et al., 2013). Thus, effective mathematics instruction in these fields involves a lot of object manipulation and visualization (Kirby & Boulter, 1999), and mentally imagining these phenomena in class may draw on exactly the same components skills as imagining the rotation of block figures or flat paper being folded into 3-D shapes. Perhaps the observed lower impact of spatial skills such as mental rotation on algebra compared with geometry and calculus is due to the fact that algebra (especially solving equations) is not as dependent on visualization and rotation of objects or figures (Battista, 1981; Weckbacher & Okamoto, 2014). As previously mentioned, the graphing functions component of algebra may be more linked to 3-D spatial processing but this connection has not yet been tested.

The mechanism by which VSWM impacts mathematics performance is that VSWM capacity is thought to be a "mental blackboard" on which operations are carried out with the help of internal visual imagery (Heathcote, 1994). The connection between VSWM and mental arithmetic is well established (Trbovich & LeFevre, 2003) and, while not necessarily the focus, mental arithmetic certainly occurs in higher-level mathematics. Ashcraft (1996) argues that VSWM is necessary for success in math because one must accurately perceive the visuospatial location of digits and variables within mathematics problems in order to solve them. Perhaps this explains the potentially greater impact of VSWM in algebra compared with geometry – the symbolic nature of algebraic equations may require more processing of numerical and variable locations, operations, and mental arithmetic. The role of VSWM has not yet been tested in calculus, but it could be predicted that students with greater VSWM capacities should have greater success with the symbolic, algebraic components in calculus as well.

It could also be argued that VSWM is where mental rotation and other visuospatial processing takes place (Heathcote, 1994), so limited VSWM necessarily restricts the spatial processing that can occur, regardless of individuals' skill with particular types of processing. However, for geometry, calculus, and some component skills in algebra (and likely other facets of algebra that are yet untested), 3-D spatial visualization may mediate the relation between working memory and math achievement (Tolar et al., 2009). Further research is certainly necessary to tease apart the relations between these two key spatial variables.

Regardless of the specific spatial skills that are influential for mathematics success, there is one other important mechanism to consider. This stems from the fact that students who do not have strong spatial skills perceive themselves to be poorer in math – even if they don't actually earn lower math scores (Weckbacher & Okamoto, 2014). Countless studies have shown that believing you will not succeed in mathematics leads to failure in mathematics, while liking or feeling you are competent in math leads to success (e.g., Eccles et al., 1983; Elliot & Church, 1997). If deficits in spatial skills cause students to doubt their competence, they are unlikely to succeed in or to pursue further study of mathematics. To date, this has only been tested explicitly with mental rotation; however, it is conceivable that the same kind of effect would be found for deficits in other spatial skills, perhaps especially VSWM as math anxiety/worry has been shown to be particularly problematic for students with low working memory (Ashcraft & Kirk, 2001).

Future Directions

It is now well established that spatial skills predict numerical skills, cross-sectionally and longitudinally, across a wide variety of age spans and using appropriate statistical controls. However, although this accomplishment is a solid one, leveraging it in education requires further work. We need to move beyond correlational analyses to evaluation of causal effects. The most obvious experiments involve intervening to improve spatial skills, and we know we can do so effectively, although it would be nice to know more about best methods, necessary duration, and other parameters of training (Uttal et al., 2013). Showing transfer to numerical skills may be a challenge, however; existing studies have shown a mixed bag of positive and negative results. One fear is that changes may be only local or at best only moderately generalizable, as research attempting to increase performance on various cognitive tasks by training working memory has arguably shown (Shipstead, Redick, & Engle, 2012).

We may be able to improve training experiments by probing more deeply into the nature of spatial–numerical linkages. As discussed in this chapter, there are many different spatial skills as well as many different mathematical operations taught at widely different ages. So, for example, it is possible that effects may vary with children's age and whether they are learning a new mathematical concept or operation or practicing one already acquired. Novice learners might rely on spatial representations to aid them in acquiring new numerical concepts (Jordan et al., 2008; McKenzie et al., 2003; Uttal & Cohen, 2012) but spatial representations might become less critical as children acquire domain-specific knowledge (e.g., memorized arithmetic facts) and algorithms for solving problems. This phenomenon has been observed among adults, for whom spatial skills are strongly related to STEM performance among novices but less so among experts, who come to rely on more knowledge-based, verbal, and analytical strategies (Hambrick et al., 2012; Stieff, 2007; Uttal & Cohen, 2012).

However, there is some contrary evidence. For example, Mix and colleagues (2016) performed an analysis in which they divided mathematics tests into those covering familiar versus novel content at each grade level and found few clear patterns of spatial predictors (although block design did predict a better grasp of novel content at all three grade levels). Furthermore, recall that Amalric and Dehaene (2016) found a great deal of overlap between the brain areas used for spatial and mathematical processing, even in expert mathematicians. Along similar lines, one study found that VSWM predicts arithmetic performance among younger children (ages 6–7, the age at which arithmetic is first introduced in school) but not older children (ages 8–9) (McKenzie et al., 2003). But another study found that VSWM predicted mathematics performance at kindergarten, 3rd grade, and 6th grade, and indeed was the strongest spatial predictor at 6th grade (Mix et al., 2016). Thus, caution is warranted about the idea that spatial thinking is most important for initial mathematics learning.

There are other versions of the general hypothesis that we need to get more specific about in terms of the nature of the linkage between spatial and mathematical cognition – for example, the work on spatial scaling and proportional reasoning discussed earlier in this chapter or the possibility that a spatial turn of mind is most important because it suggests strategies such as sketching during problem-solving that are known to be helpful (Miller-Cotto et al., under review). Evaluating these ideas will be the challenge for the next decade.

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